Estimates of tempered stable densities

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Abstract

Estimates of densities of convolution semigroups of probability measures are given under specific assumptions on the corresponding Lévy measure and the Lévy–Khinchin exponent. The assumptions are satisfied, e.g., by tempered stable semigroups of J. Rosiński.

1 Introduction and main results

Let ν be a symmetric Lévy measure on $\mathbb{R}^d$ where $d \in \{1, 2, 3, \ldots\}$.

We consider the convolution semigroup of probability measures $\{P_t, t \geq 0\}$ with the Fourier transform $\mathcal{F}(P_t)(\xi) = \exp(-t\Phi(\xi))$, where

$$\Phi(\xi) = \int (1 - \cos(\xi \cdot y))\, \nu(dy).$$

The semigroup determines the stochastic Lévy process (X_t, P^x) on $\mathbb{R}^d$ with the generating triplet $(0, \nu, 0)$ (we use the terminology of [22]) and transition probabilities $P(t, x, A) = P_t(A - x)$. If

$$\nu(D) = \int_{\mathbb{S}} \int_0^\infty \mathbf{1}_D(s\theta) s^{-1-\alpha} ds \mu(d\theta),$$

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where $\alpha \in (0,2)$ and μ is a bounded measure on $\mathbb{S} = \{x \in \mathbb{R}^d \ : |x| = 1\}$, then (X_t, P^x) is called the α*–stable Lévy process.* If μ is nondegenerate, i.e., if there is no proper linear subspace M of $\mathbb{R}^d$ such that $\mathrm{supp}(\mu) \subset M$, then the α–stable symmetric measure P_t is absolutely continuous for every $t > 0$ and the corresponding density p_t is smooth, bounded and it has the *scaling property*: $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x)$.

Stable processes are important tool in theoretical probability, physics and finance and their asymptotic properties are subject of interest of many papers. W.E. Pruitt and S.J. Taylor investigated in [19] multivariate stable densities in the general setting. They obtained the estimate $p_1(x) \le c(1+|x|)^{-1-\alpha}$ by Fourier-analytic methods. Indeed such a decay can be obtained if the spectral measure μ has an atom (see the estimates from below in [12] and [13]). In the well–known case of the rotation invariant α–stable process with uniform μ we however have $p_1(x) \approx (1+|x|)^{-d-\alpha}$ (see [26]). Using the perturbation formula P. Głowacki and W. Hebisch proved in [8] and [9] that if μ has a bounded density, g_μ, with respect to the surface measure on $\mathbb{S}$ then $p_1(x) \le c(1+|x|)^{-d-\alpha}$. When g_μ is continuous on $\mathbb{S}$ we even have $\lim_{r\to\infty} r^{d+\alpha} p_1(r\theta) = c g_\mu(\theta)$, $\theta \in \mathbb{S}$ and if $g_\mu(\theta) = 0$ then additionally $\lim_{r\to\infty} r^{d+2\alpha} p_1(r\theta) = c_\theta > 0$, which was proved by J. Dziubański in [7]. A. Zaigraev in [25] obtained further asymptotic expansions of the α–stable density for sufficiently regular g_μ.

More recent asymptotic results for stable Lévy processes are given in papers [24] and [2]. In particular if for some $\gamma \in [1,d]$ the measure ν is a γ–measure on $\mathbb{S}$, i.e.,

$$\nu(B(x,r)) \le c r^\gamma \quad \text{for every} \quad x \in \mathbb{S},\, r \le 1/2,$$

or equivalently

$$\mu(B(\theta,r) \cap \mathbb{S}) \le c r^{\gamma-1}, \quad \theta \in \mathbb{S},\, r \le 1/2,$$

then we have

$$p_1(x) \le c(1+|x|)^{-\alpha-\gamma}, \quad x \in \mathbb{R}^d.$$

By scaling $p_t(x) \le c t^{-d/\alpha}(1+t^{-1/\alpha}|x|)^{-\alpha-\gamma}$ for every $t > 0$. It follows also from [24, Theorem 1.1] that if for some $\theta_0 \in \mathbb{S}$ we have

$$\mu(B(\theta_0,r) \cap \mathbb{S}) \ge c r^{\gamma-1}, \quad r \le 1/2,$$

then

$$p_1(r\theta_0) \ge c(1+r)^{-\alpha-\gamma}, \quad r > 0.$$

Our main goal is to extend the estimates to more general class of semigroups and processes. The obtained below results cover a wide class of examples which we discuss in more detail in the Section 5.

We fix the constant $\alpha \in (0,2)$ and we always assume that there exists a positive c such that

$$\Phi(\xi) \geq c|\xi|^\alpha \text{ for } |\xi| > 1. \tag{1}$$

Note that (1) is satisfied, e.g., if we have

$$\int_{|y|\leq r} |\theta \cdot y|^2 \nu(dy) \geq c r^{2-\alpha} \quad \text{for} \quad r \leq 1,\, \theta \in \mathbb{S}.$$

It follows from (1) that the measures P_t are absolutely continuous with respect to the Lebesgue measure and their densities p_t are smooth and bounded.

All the sets, functions and measures considered in the sequel will be Borel.

Theorem 1 *Let there exist constants $c > 0$, $\gamma \in [1,d]$, $K \in [1,\infty)$, a bounded measure μ on $\mathbb{S}$, and a bounded nonincreasing function $\bar{q} : (0,\infty) \to (0,\infty)$, such that*

$$\nu(A) \leq \int_{\mathbb{S}} \int_0^\infty \mathbf{1}_A(s\theta) s^{-1-\alpha} \bar{q}(s) ds \mu(d\theta), \quad A \subset \mathbb{R}^d, \tag{2}$$

$$\bar{q}(s) \leq K \bar{q}(2s), \quad s > 0, \tag{3}$$

and

$$\mu(B(\theta, r) \cap \mathbb{S}) \leq c r^{\gamma-1}, \quad \theta \in \mathbb{S},\, r < 1/2. \tag{4}$$

Then there exists a constant C such that

$$\begin{aligned} p_t(x) &\leq C \min\left(t^{-d/\alpha}, t^{1+\frac{\gamma-d}{\alpha}} |x|^{-\alpha-\gamma}\, \bar{q}(|x|)\right) \\ &\approx t^{-d/\alpha}\left(1 + t^{-1/\alpha}|x|\right)^{-\alpha-\gamma} \bar{q}(|x|), \quad x \in \mathbb{R}^d,\, t \in (0,1). \end{aligned} \tag{5}$$

The *doubling property* (3) is equivalent to the following: there exist $c > 0$ and $\eta \geq 0$ such that

$$\frac{\bar{q}(r)}{\bar{q}(R)} \leq c \left(\frac{r}{R}\right)^{-\eta}, \quad 0 < r \leq R. \tag{6}$$

The typical examples of $\bar{q}$ are $\bar{q}(s) = (1+s)^{-a}$, or $\bar{q}(s) = (\log(e+s))^a (1+s)^{-ma}$, for $a \geq 0$ and $m > 1$.

We note that if (2), (3) and (4) hold then we have

$$\nu(B(x,r)) \leq c r^{\gamma} |x|^{-\alpha-\gamma} \bar{q}(|x|), \quad r < \frac{1}{2}|x|,\, x \in \mathbb{R}^d \setminus \{0\}, \tag{7}$$

and

$$\nu(B(0,r)^c) \leq c r^{-\alpha} \bar{q}(r), \quad r \in (0,\infty). \tag{8}$$

We omit the easy proof. The partial converse of Theorem 1 is given in the following theorem.

THEOREM 2 *Let there exist a set $A \subset \mathbb{R}^d$, constants $\gamma \in [1,d]$, $c_1, c_2 > 0$ and a bounded function $\underline{q} : (0,\infty) \to (0,\infty)$ such that*

$$\nu(B(x,r)) \geq c_1 r^{\gamma} |x|^{-\alpha-\gamma} \underline{q}(|x|), \quad x \in A,\, r > 0, \tag{9}$$

$$\nu(B(0,r)^c) \leq c_2 r^{-\alpha}, \quad r \in (0,1). \tag{10}$$

Then there exists a constant C such that

$$p_t(x) \geq C \min\left(t^{-d/\alpha}, t^{1+\frac{\gamma-d}{\alpha}} |x|^{-\alpha-\gamma}\, \underline{q}(|x|)\right), \quad x \in A,\, t \in (0,1). \tag{11}$$

The mild assumptions on the function $\underline{q}$ allow to use Theorem 2 for a wide class of processes. An important example is the relativistic α–stable process (see [21, 11]) with $\Phi(\xi) = (|\xi|^2+1)^{\alpha/2} - 1$. It follows from [11] that in this case (9) and (10) hold with $\gamma = d$, $A = \mathbb{R}^d$, and $\underline{q}(s) = (1+s)^{\frac{d+\alpha-1}{2}} e^{-2s}$, and we get $p_t(x) \geq c \min\left(t^{-d/\alpha}, t|x|^{-\alpha-d}(1+|x|)^{\frac{d+\alpha-1}{2}} e^{-2|x|}\right)$ for $x \in \mathbb{R}^d$ and $t \in (0,1)$. We discuss this example in detail in Section 5.

The above theorems hold for small times t and below we consider the case of large t. We assume still (1) which guarantees in particular the existence and smoothness of densities. However the behaviour of p_t at the origin for large t depends on the asymptotic of $\Phi(\xi)$ for small ξ (see Lemma 8) where Φ may decay faster then $|\xi|^\alpha$. Hence we strengthen below our assumptions.

THEOREM 3 *Let (2), (3) and (4) hold and let there exist constants c and $\beta \in [\alpha, 2]$ such that*

$$\int_1^\infty s^{\beta-\alpha-1} \bar{q}(s) < \infty, \tag{12}$$

and

$$\Phi(\xi) \geq c|\xi|^\beta, \quad |\xi| \leq 1. \tag{13}$$

Then there exists a constant C such that

$$p_t(x) \le C \min\left(t^{-d/\beta}, t^{1+\frac{\gamma-d}{\beta}}|x|^{-\alpha-\gamma}\bar{q}(|x|)\right), \quad x \in \mathbb{R}^d,\ t \in (1,\infty). \tag{14}$$

THEOREM 4 *Let (9) hold and let there exist constant $\beta \in [\alpha, 2]$ and c such that*

$$\nu(B(0,r)^c) \le c r^{-\beta}, \quad r \in (0,\infty), \tag{15}$$

$$c^{-1}|\xi|^\beta \le \Phi(\xi) \le c|\xi|^\beta, \quad |\xi| \le 1. \tag{16}$$

Then there exists a constant C such that

$$p_t(x) \ge C \min(t^{-d/\beta}, t^{1+\frac{\gamma-d}{\beta}}|x|^{-\alpha-\gamma}\underline{q}(|x|)), \quad x \in A,\ t \in (1,\infty). \tag{17}$$

We like to mention related recent results. Z.-Q. Chen and T. Kumagai in [4] and [5] investigate the case of symmetric jump–type Markov processes on some class of metric measure spaces with jump kernels which are not translation invariant. They obtain estimates of the densities in small time (see Theorem 1.2 in [5]) which are similar to given above. However the jump kernels in [4] and [5] are assumed to have densities (which corresponds to $\gamma = d$ in our setting) comparable with certain rotation invariant functions. Here the assumptions (2) and (4) include the isotropic estimates only from above and in Theorems 2 and 4 the estimate (9) can depend on a chosen direction so we allow for more anisotropy here. The results for large t are restricted in [5] to the case of jump densities which decay in infinity not faster then $|x|^{-2-d}$.

The asymptotics of the densities of the truncated stable–like processes have been investigated in [3] by Z.-Q. Chen, P. Kim and T. Kumagai. K. Bogdan and T. Jakubowski in [1] obtained estimates of the heat kernels of the fractional Laplacian perturbed by gradient operators. The derivatives of the stable densities have been considered in [17] and [23].

J. Picard [18] studied the density in small time for jump processes obtaining the estimates for solutions of certain stochastic differential equation driven by stable Lévy process. Here we improve the methods used in [18] and [2] and extend the results to a wide class of Lévy non–stable processes. Our assumptions are satisfied, e.g., by the tempered stable processes of J. Rosiński ([20]) and the layered stable processes of C. Houdré and R. Kawai ([14]).

The paper is organized as follows. In Section 2 we investigate densities of the semigroup determinated by the truncated Lévy measure $\mathbf{1}_{B(0,\varepsilon)}\nu$ where ε is comparable with $t^{1/\alpha}$ for small t and $t^{1/\beta}$ for large t. In Section 3 we consider $\mathbf{1}_{B(0,\varepsilon)^c}\nu$ which is a bounded measure and therefore the corresponding

semigroup is given by an exponential formula. We collect these results in Section 4 and prove the main theorems. Examples are given in Section 5.

We use c, C (with subscripts) to denote finite positive constants which depend only on the measure μ, the functions $\bar{q}, \underline{q}$, constants γ, α, β, and the dimension d. Any *additional* dependence is explicitly indicated by writing, e.g., $c = c(n)$. The value of c, when used without subscripts, may change from place to place. We write $f \approx g$ to indicate that there is a c such that $c^{-1} f \le g \le cf$.

2 Local part

For $\varepsilon > 0$ we denote $\tilde{\nu}_\varepsilon = \mathbf{1}_{B(0,\varepsilon)}\nu$. We consider the corresponding semigroup of measures $\{\tilde{P}^\varepsilon_t,\ t \ge 0\}$ such that

$$\mathcal{F}(\tilde{P}^\varepsilon_t)(\xi) = \exp\left(t \int (\cos(\xi \cdot y) - 1)\tilde{\nu}_\varepsilon(dy)\right), \quad \xi \in \mathbb{R}^d. \tag{18}$$

By (1) we get

$$\begin{aligned}
\mathcal{F}(\tilde{P}^\varepsilon_t)(\xi) &= \exp\left(-t \int_{|y|<\varepsilon} (1 - \cos(y \cdot \xi))\nu(dy)\right) \\
&= \exp\left(-t\left(\Phi(\xi) - \int_{|y|\ge\varepsilon} (1 - \cos(y \cdot \xi))\nu(dy)\right)\right) \\
&\le \exp(-t\Phi(\xi)) \exp(2t\nu(B(0,\varepsilon)^c)) \\
&\le \exp(-ct|\xi|^\alpha) \exp(2t\nu(B(0,\varepsilon)^c)), \quad |\xi| > 1. \qquad (19)
\end{aligned}$$

From (19), [15, Theorem 3.7.13] and the multivariable version of Faa di Bruno's formula (see, e.g., [6]) it follows that for every $\varepsilon > 0$ and $t > 0$ the measure $\tilde{P}^\varepsilon_t$ is absolutely continuous with respect to the Lebesgue measure and its density $\tilde{p}^\varepsilon_t$ belongs to the Schwarz space of smooth rapidly decreasing functions.

We will often use $\tilde{P}^\varepsilon_t$ and $\tilde{p}^\varepsilon_t$ with $\varepsilon = t^{1/\alpha}$. For simplification we will denote

$$\tilde{P}_t = \tilde{P}^{t^{1/\alpha}}_t \text{ and } \tilde{p}_t = \tilde{p}^{t^{1/\alpha}}_t.$$

Since $\tilde{p}_t$ belongs to the Schwarz space we certainly have that $\tilde{p}_t(y) \le c(t,m)(1 + |y|)^{-m}$ for every $m > 0$ and $t > 0$. We improve the estimates in the following two Lemmas.

Lemma 1 *If we have*

$$\nu(B(0,r)^c) \le c r^{-\alpha}, \ \textit{for } r \in (0,\infty), \tag{20}$$

then for every $n \in \mathbb{N}$ there exists a constant $c_n = c(n)$ such that

$$\int_{\mathbb{R}^d} |y|^{2n} \tilde{P}_t(dy) \le c_n t^{2n/\alpha}, \quad t > 0. \tag{21}$$

Proof. Let $f_\varepsilon(\xi) = \int_{\mathbb{R}^d}(1-\cos(y\cdot\xi))\tilde{\nu}_\varepsilon(dy)$. For every $l \in \mathbb{N}$ and $k \in \{1,\ldots,d\}$ we have

$$\frac{\partial^{2l} f_\varepsilon}{\partial \xi_k^{2l}}(0) = (-1)^{l+1}\int_{\mathbb{R}^d} y_k^{2l}\tilde{\nu}_\varepsilon(dy), \tag{22}$$

and

$$\frac{\partial^{2l-1} f_\varepsilon}{\partial \xi_k^{2l-1}}(0) = 0. \tag{23}$$

We use the Faa di Bruno's formula (see [6]), (22) and (23) to obtain

$$\begin{aligned} \frac{\partial^{2n}}{\partial \xi_k^{2n}}\mathcal{F}(\tilde{P}_t^\varepsilon)(0) &= (2n)!\exp(-tf_\varepsilon(0))\sum_{j=1}^{2n}\sum_{\pi(2n,j)}\prod_{l=1}^{2n}\frac{\left(\frac{\partial^l(-tf_\varepsilon)}{\partial\xi_k^l}(0)\right)^{\lambda_l}}{(\lambda_l!)(l!)^{\lambda_l}} \\ &= \sum_{j=1}^{2n}\sum_{\pi'(2n,j)} t^j \prod_{l=1}^{n}\frac{(-1)^n(2n)!}{(\lambda_{2l}!)((2l)!)^{\lambda_{2l}}}\left[\int_{\mathbb{R}^d} y_k^{2l}\tilde{\nu}_\varepsilon(dy)\right]^{\lambda_{2l}} \end{aligned} \tag{24}$$

where

$$\pi(2n,j) = \{(\lambda_1,\ldots,\lambda_{2n}) : \ \lambda_l \in \mathbb{N}_0, \sum_{l=1}^{2n}\lambda_l = j, \sum_{l=1}^{2n} l\lambda_l = 2n\}$$

and

$$\pi'(2n,j) = \pi(2n,j) \cap \{(\lambda_1,\ldots,\lambda_{2n}) : \ \lambda_l \in \mathbb{N}_0, \ \lambda_1 = \lambda_3 = \ldots = \lambda_{2n-1} = 0\}.$$

For every $n \in \mathbb{N}$ and $k \in \{1,\ldots,d\}$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} y_k^{2n}\tilde{P}_t^\varepsilon(dy) &= \frac{1}{i^{2n}}\left[\frac{\partial^{2n}}{\partial\xi_k^{2n}}\mathcal{F}(\tilde{P}_t^\varepsilon)(\xi)\right]_{\xi=0} \\ &= (-1)^n\left[\frac{\partial^{2n}}{\partial\xi_k^{2n}}\mathcal{F}(\tilde{P}_t^\varepsilon)(\xi)\right]_{\xi=0}, \end{aligned}$$

and this and (24) yield

$$\int_{\mathbb{R}^d} |y|^{2n}\tilde{P}^\varepsilon_t(dy) \;\leq\; d^{n-1}\int_{\mathbb{R}^d}(y_1^{2n}+\ldots+y_d^{2n})\tilde{P}^\varepsilon_t(dy)$$

$$(25)\qquad = d^{n-1}\sum_{k=1}^{d}\sum_{j=1}^{2n}\sum_{\pi'(2n,j)} t^j \prod_{l=1}^{n}\frac{(2n)!}{(\lambda_{2l}!)((2l)!)^{\lambda_{2l}}}\left[\int_{\mathbb{R}^d} y_k^{2l}\tilde{\nu}_\varepsilon(dy)\right]^{\lambda_{2l}}.$$

For every $k\in\{1,\ldots,d\}$ and $l\in\mathbb{N}$ by (20) we have

$$\begin{aligned}
\int_{\mathbb{R}^d} y_k^{2l}\tilde{\nu}_\varepsilon(dy) &= \sum_{j=0}^{\infty}\int_{2^{-j-1}\varepsilon\leq|y|<2^{-j}\varepsilon} y_k^{2l}\tilde{\nu}_\varepsilon(dy)\\
&\leq \sum_{j=0}^{\infty}(2^{-j}\varepsilon)^{2l}\nu(B(0,2^{-j-1}\varepsilon)^c)\\
&\leq \varepsilon^{2l}\sum_{j=0}^{\infty}(2^{2l})^{-j}c(2^{-j-1}\varepsilon)^{-\alpha} = c\varepsilon^{2l-\alpha}.
\end{aligned}$$

and by (25) we obtain

$$\int_{\mathbb{R}^d} |y|^{2n}\tilde{P}^\varepsilon_t(dy) \;\leq\; \varepsilon^{2n}d^{n-1}\sum_{k=1}^{d}\sum_{j=1}^{2n} c(j,n)t^j\varepsilon^{-j\alpha}.$$

Taking $\varepsilon = t^{1/\alpha}$ we get (21). $\square$

LEMMA 2 *If (20) is satisfied then for every $m\geq 1$ there exists a constant $c_m = c(m)$ such that*

$$(26)\qquad \tilde{p}_t(y) \leq c(m)t^{-d/\alpha}(1+t^{-1/\alpha}|y|)^{-m},\quad y\in\mathbb{R}^d,\ t\in(0,1).$$

Proof. From (20) and (19) we get $\mathcal{F}(\tilde{p}_t)(\xi)\leq c_1\exp(-c_2t|\xi|^\alpha)$ for $|\xi|>1$ and $\mathcal{F}(\tilde{p}_t)(\xi)\leq 1$ for $|\xi|\leq 1$. We denote $g_t(y) = t^{d/\alpha}\tilde{p}_t(t^{1/\alpha}y)$. For every $j\in\{1,\ldots,d\}$ we obtain

$$\begin{aligned}
\left|\frac{\partial g_t}{\partial y_j}(y)\right| &= \left|t^{d/\alpha}(2\pi)^{-d}\int_{\mathbb{R}^d}(-i)t^{1/\alpha}\xi_j e^{-it^{1/\alpha}y\cdot\xi}\mathcal{F}(\tilde{p}_t)(\xi)d\xi\right|\\
&\leq ct^{\frac{1+d}{\alpha}}\left(\int_{|\xi|\leq 1}|\xi_j|d\xi + \int_{|\xi|>1}|\xi_j|e^{-c_2t|\xi|^\alpha}d\xi\right)\\
&= ct^{\frac{1+d}{\alpha}} + c\int_{|u|>t^{1/\alpha}}|u_j|e^{-c_2|u|^\alpha}du\\
&\leq c,\quad y\in\mathbb{R}^d,\ t\in(0,1). \qquad (27)
\end{aligned}$$

Similarly we get $g_t(y) \le c$ for $y \in \mathbb{R}^d$, $t \in (0,1)$. By Lemma 1 we obtain $\int_{\mathbb{R}^d} |y|^{2n} g_t(y)dy \le c(n)$, and also $\int_{\mathbb{R}^d} |y|^{2n-1} g_t(y)dy \le \int_{|y|\le 1} g_t(y)dy + \int_{|y|>1} |y|^{2n} g_t(y)dy \le 1 + c(n)$ and by [18, Lemma 9] for every $m \ge 1$ we get

$$g_t(y) \le c(m)(1+|y|)^{-m}, \quad y \in \mathbb{R}^d,\, t \in (0,1),$$

which clearly yields (26). $\square$

We prove now the estimate of $\tilde{p}_t^{\varepsilon}(y)$ from below for small y, $\varepsilon = at^{1/\alpha}$ and $a \in (0,1]$.

Lemma 3 *If (10) holds then there exist constants c, c_1, c_2 such that*

$$\tilde{p}_t^{at^{1/\alpha}}(y) \ge ct^{-d/\alpha}, \tag{28}$$

provided $|y| \le c_1 e^{-c_2 a^{-\alpha}} t^{1/\alpha}$, $t \in (0,1)$, and $a \in (0,1]$.

Proof. It follows from (10) that

$$\begin{aligned}
\Phi(\xi) &\le \frac{1}{2}\int_{|y|\le 1/|\xi|} |\xi \cdot y|^2 \nu(dy) + 2\nu(B(0,1/|\xi|)^c) \\
&\le \frac{1}{2}|\xi|^2 \sum_{j=0}^{\infty} \int_{\frac{2^{-j-1}}{|\xi|}<|y|\le\frac{2^{-j}}{|\xi|}} |y|^2 \nu(dy) + 2\nu(B(0,1/|\xi|)^c) \\
&\le \frac{1}{2}|\xi|^2 \sum_{j=0}^{\infty} 2^{-2j}|\xi|^{-2} \nu(B(0,\frac{2^{-j-1}}{|\xi|})^c) + 2\nu(B(0,1/|\xi|)^c) \\
&\le c|\xi|^\alpha, \quad |\xi| > 1. \qquad (29)
\end{aligned}$$

Let $g_t(y) = t^{d/\alpha} \tilde{p}_t^{at^{1/\alpha}}(t^{1/\alpha} y)$. We have

$$\mathcal{F}(\tilde{p}_t^{at^{1/\alpha}})(\xi) \ge \mathcal{F}(p_t)(\xi), \quad \xi \in \mathbb{R}^d,\, t > 0,$$

and this and (29) yield

$$\begin{aligned}
g_t(0) = t^{d/\alpha}\tilde{p}_t^{at^{1/\alpha}}(0) &\ge t^{d/\alpha} p_t(0) \\
&= t^{d/\alpha}(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Phi(\xi)} d\xi \\
&\ge t^{d/\alpha}(2\pi)^{-d} \int_{|\xi|\ge 1} e^{-ct|\xi|^\alpha} d\xi \\
&\ge c_0 > 0, \quad t \in (0,1). \qquad (30)
\end{aligned}$$

For every $j \in \{1, \ldots, d\}$ by (19) and (10) we get

$$\begin{aligned} \left|\frac{\partial g_t}{\partial y_j}(y)\right| &= \left|t^{d/\alpha}(2\pi)^{-d}\int_{\mathbb{R}^d}(-i)t^{1/\alpha}\xi_j e^{-it^{1/\alpha}y\cdot\xi}\mathcal{F}(\tilde{p}_t^{at^{1/\alpha}})(\xi)d\xi\right| \\ &\le c_3 e^{c_2 a^{-\alpha}} t^{\frac{1+d}{\alpha}}\left(\int_{|\xi|\le 1}|\xi_j|d\xi + \int_{|\xi|>1}|\xi_j|e^{-c_4 t|\xi|^\alpha}d\xi\right) \\ &\le c_5 e^{c_2 a^{-\alpha}}, \quad y \in \mathbb{R}^d,\ t \in (0,1). \end{aligned}$$

It follows that

$$g_t(y) \ge c_0 - dc_5 e^{c_2 a^{-\alpha}}|y| \ge c_0/2,$$

provided $|y| \le c_1 e^{-c_2 a^{-\alpha}}$, for some c_1 which clearly yields (28). □

3 Bounded part

For a measure λ on $\mathbb{R}^d$, $|\lambda|$ denotes its total mass. When $|\lambda| < \infty$ and $n = 1, 2, \ldots$ we let λ^{n*} denote the n-fold convolution of λ with itself:

$$\lambda^{n*}(f) = \int f(x_1 + x_2 + \ldots + x_n)\lambda(dx_1)\lambda(dx_2)\ldots\lambda(dx_n)\,.$$

We also let $\lambda^{0*} = \delta_0$, the evaluation at 0.

For $\varepsilon > 0$ we denote $\bar{\nu}_\varepsilon = \mathbf{1}_{B(0,\varepsilon)^c}\nu$. We consider the corresponding semigroup of measures $\{\bar{P}_t^\varepsilon,\ t \ge 0\}$ such that

$$\mathcal{F}(\bar{P}_t^\varepsilon)(\xi) = \exp\left(t\int(\cos(\xi\cdot y) - 1)\bar{\nu}_\varepsilon(dy)\right), \quad \xi \in \mathbb{R}^d\,. \tag{31}$$

Note that

$$\begin{aligned} \bar{P}_t^\varepsilon &= \exp(t(\bar{\nu}_\varepsilon - |\bar{\nu}_\varepsilon|\delta_0)) = \sum_{n=0}^\infty \frac{t^n\,(\bar{\nu}_\varepsilon - |\bar{\nu}_\varepsilon|\delta_0))^{n*}}{n!} \\ &= e^{-t|\bar{\nu}_\varepsilon|}\sum_{n=0}^\infty \frac{t^n\bar{\nu}_\varepsilon^{n*}}{n!}\,, \quad t \ge 0\,, \end{aligned} \tag{32}$$

and

$$P_t = \tilde{P}_t^\varepsilon * \bar{P}_t^\varepsilon\,, \quad t \ge 0.$$

Of course,

$$p_t = \tilde{p}_t^\varepsilon * \bar{P}_t^\varepsilon\,, \quad t > 0. \tag{33}$$

LEMMA 4 *If (2), (3) and (4) are satisfied then there exists a constant c such that*

$$\bar{\nu}_\varepsilon^{n*}(B(x,r)) \le c^n r^\gamma (\varepsilon^{-\alpha}\bar{q}(\varepsilon))^{n-1}|x|^{-\alpha-\gamma}\bar{q}(|x|), \tag{34}$$

provided $x \in \mathbb{R}^d \setminus \{0\}$, $\varepsilon > 0$, $n \in \mathbb{N}$, and $r \le \max(\frac{\varepsilon}{3}, \frac{|x|}{5^n})$.

Proof. We proceed by induction. Note that (34) for $n = 1$ holds by (7) and (8). Let c_0 and n be such that (34) is satisfied with $c = c_0$. We first assume that $r \le \varepsilon/3$. By (7) we have

$$\begin{aligned}
\bar{\nu}_\varepsilon^{(n+1)*}(B(x,r)) &= \int\limits_{|x-y|>2\varepsilon/3} \bar{\nu}_\varepsilon(B(x-y,r))\bar{\nu}_\varepsilon^{n*}(dy) \\
&\le \int\limits_{|x-y|>2\varepsilon/3} \nu(B(x-y,r))\bar{\nu}_\varepsilon^{n*}(dy) \\
&\le c_1 r^\gamma \int\limits_{|x-y|>2\varepsilon/3} |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy)
\end{aligned}$$

(note that $r < |x-y|/2$ provided $|x-y| > 2\varepsilon/3$). Now let $\varepsilon/3 < r \le |x|/5^{n+1}$. Then $2r + \varepsilon < |x|/5^n$ and by induction and (8)

$$\begin{aligned}
\int\limits_{|x-y|<2r+\varepsilon} \bar{\nu}_\varepsilon(B(x-y,r))\bar{\nu}_\varepsilon^{n*}(dy) &\le |\bar{\nu}_\varepsilon|\bar{\nu}_\varepsilon^{n*}(B(x,2r+\varepsilon)) \\
&\le c_2\varepsilon^{-\alpha}\bar{q}(\varepsilon)c_0^n(2r+\varepsilon)^\gamma(\varepsilon^{-\alpha}\bar{q}(\varepsilon))^{n-1}|x|^{-\alpha-\gamma}\bar{q}(|x|) \\
&\le c_0^n c_3 r^\gamma (\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n |x|^{-\alpha-\gamma}\bar{q}(|x|),
\end{aligned}$$

for some c_3; and by (7) we get

$$\begin{aligned}
\int\limits_{|x-y|\ge 2r+\varepsilon} \bar{\nu}_\varepsilon(B(x-y,r))\bar{\nu}_\varepsilon^{n*}(dy) &\le \int\limits_{|x-y|\ge 2r+\varepsilon} \nu(B(x-y,r))\bar{\nu}_\varepsilon^{n*}(dy) \\
&\le \int\limits_{|x-y|\ge 2r+\varepsilon} c_1 r^\gamma |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy) \\
&\le c_1 r^\gamma \int\limits_{|x-y|>2\varepsilon/3} |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy).
\end{aligned}$$

From the above we have

$$
\begin{aligned}
(35) \qquad \bar{\nu}_\varepsilon^{(n+1)*}(B(x,r)) &\leq c_1 r^\gamma \int\limits_{|x-y|>2\varepsilon/3} |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy) \\
&\quad + c_0^n c_3 r^\gamma (\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n |x|^{-\alpha-\gamma}\bar{q}(|x|),
\end{aligned}
$$

for all $0 < r \leq \max(\varepsilon/3, |x|/5^{n+1})$.

Let $L_\varepsilon = \left\lfloor \log_5(\frac{3|x|}{2\varepsilon}) \right\rfloor$. If $2\varepsilon/3 < |x|/5^n$ then we get by (6) and induction

$$
\begin{aligned}
&\int\limits_{2\varepsilon/3<|x-y|<|x|/5^n} |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy) \\
&\leq \bar{q}(2\varepsilon/3)\sum_{k=n}^{L_\varepsilon} \int\limits_{|x|/5^{k+1}\leq|x-y|<|x|/5^k} |x-y|^{-\alpha-\gamma}\bar{\nu}_\varepsilon^{n*}(dy) \\
&\leq \bar{q}(2\varepsilon/3)\sum_{k=n}^{L_\varepsilon} (5^{k+1})^{\alpha+\gamma}|x|^{-\alpha-\gamma}\bar{\nu}_\varepsilon^{n*}(B(x,|x|/5^k)) \\
&\leq c_0^n \bar{q}(2\varepsilon/3)5^{\alpha+\gamma}(\varepsilon^{-\alpha}\bar{q}(\varepsilon))^{n-1}|x|^{-2\alpha-\gamma}\bar{q}(|x|)\sum_{k=1}^{L_\varepsilon} 5^{k\alpha} \\
&\leq c_0^n c_4 (\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n |x|^{-\alpha-\gamma}\bar{q}(|x|)\,.
\end{aligned}
$$

Also, by (6) and (8)

$$
\begin{aligned}
\int\limits_{|x-y|\geq|x|/5^n} |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy) &\leq (5^{\alpha+\gamma})^n|x|^{-\alpha-\gamma}c_5\bar{q}(|x|)5^{n\eta}|\bar{\nu}_\varepsilon^{n*}| \\
&\leq c_5 c_0^n (\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n|x|^{-\alpha-\gamma}\bar{q}(|x|),
\end{aligned}
$$

by taking large c_0. We get

$$
\int\limits_{|x-y|>2\varepsilon/3} |x-y|^{-\alpha-\gamma}\bar{q}(|x-y|)\bar{\nu}_\varepsilon^{n*}(dy) \leq c_0^n(\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n(c_4+c_5)|x|^{-\alpha-\gamma}\bar{q}(|x|),
$$

and (35) yields

$$
\bar{\nu}_\varepsilon^{(n+1)*}(B(x,r)) \leq c_0^{n+1} r^\gamma (\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n |x|^{-\alpha-\gamma}\bar{q}(|x|).
$$

□

Corollary 5 *If (2), (3) and (4) are satisfied then there exists c such that*

$$
\begin{aligned}
(36) \qquad \bar{\nu}_\varepsilon^{n*}(B(x,r)) &\leq c^n(\varepsilon^{-\alpha}\bar{q}(\varepsilon))^{n-1} r^\gamma \Big(1 + \frac{\varepsilon^{-\alpha}\bar{q}(\varepsilon)}{r^{-\alpha}\bar{q}(r)}\Big)|x|^{-\alpha-\gamma}\bar{q}(|x|), \\
&\qquad x \in \mathbb{R}^d,\ r > 0,\ \varepsilon > 0,\ n \in \mathbb{N}.
\end{aligned}
$$

Proof. If $r \le |x|/5^n$ then (36) follows directly from Lemma 4. If $r > |x|/5^n$ then by (8) and (6) we have

$$\begin{aligned}\bar{\nu}_\varepsilon^{n*}(B(x,r)) &\le |\bar{\nu}_\varepsilon^{n*}| \le c^n(\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n \\ &\le c^n(\varepsilon^{-\alpha}\bar{q}(\varepsilon))^n\left(\frac{r5^n}{|x|}\right)^{\gamma+\alpha}\frac{\bar{q}(|x|)5^{n\eta}}{\bar{q}(r)}.\end{aligned}$$

□

We will denote

$$\bar{P}_t = \bar{P}_t^{t^{1/\alpha}}.$$

COROLLARY 6 *If (2), (3) and (4) are satisfied then there exists c such that*

$$\bar{P}_t(B(x,r)) \le c\frac{t}{\bar{q}(t^{1/\alpha})}r^\gamma\left(1+\frac{t^{-1}\bar{q}(t^{1/\alpha})}{r^{-\alpha}\bar{q}(r)}\right)|x|^{-\alpha-\gamma}\bar{q}(|x|)\,,\quad r>0\,,\ t>0\,,\ x\in\mathbb{R}^d.$$

Proof. Corollary 5 and (32) yield

$$\begin{aligned}\bar{P}_t(B(x,r)) &\le \sum_{n=0}^\infty \frac{t^n\bar{\nu}_{t^{1/\alpha}}^{n*}(B(x,r))}{n!} \\ &\le \sum_{n=0}^\infty \frac{c^n t(\bar{q}(t^{1/\alpha}))^{n-1}r^\gamma\left(1+\frac{t^{-1}\bar{q}(t^{1/\alpha})}{r^{-\alpha}\bar{q}(r)}\right)|x|^{-\alpha-\gamma}\bar{q}(|x|)}{n!} \\ &\le e^{c\|q\|_\infty}\frac{t}{\bar{q}(t^{1/\alpha})}r^\gamma\left(1+\frac{t^{-1}\bar{q}(t^{1/\alpha})}{r^{-\alpha}\bar{q}(r)}\right)|x|^{-\alpha-\gamma}\bar{q}(|x|).\end{aligned}$$

□

4 Proofs of main results

In the following two lemmas we investigate the transition densities at the origin. The behaviour of Φ at infinity determinates the asymptotic of p_t in small time whereas considering the rate of decay of Φ at the origin we obtain the estimates of p_t for large t.

LEMMA 7 *If (1) holds then there exists c such that*

$$p_t(x) \le ct^{-d/\alpha}, \quad x\in\mathbb{R}^d,\ t\in(0,1).$$

If (10) is satisfied then there exist constants c and δ such that

(37) $$p_t(x) \ge ct^{-d/\alpha}, \quad |x|\le\delta t^{1/\alpha},\ t\in(0,1).$$

Proof. By (1) we have

$$\begin{aligned}
p_t(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\Phi(\xi)} d\xi \\
&\le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Phi(\xi)} d\xi \\
&\le (2\pi)^{-d} \left(\int_{|\xi|\le 1} d\xi + \int_{|\xi|\ge 1} e^{-c_1 t|\xi|^\alpha} d\xi \right) \\
&= (2\pi)^{-d} \left(c_2 + t^{-d/\alpha} \int_{|u|\ge t^{1/\alpha}} e^{-c_1|u|^\alpha} du \right) \\
&\le c_3 \left(1 + t^{-d/\alpha}\right) \\
&\le 2c_3 t^{-d/\alpha}, \quad x \in \mathbb{R}^d,\ t \in (0,1).
\end{aligned}$$

The proof of (37) is analogous to the proof of Lemma 3 and therefore we omit the details. □

Lemma 8 *If (13) holds then there exists c such that*

$$p_t(x) \le ct^{-d/\beta}, \quad x \in \mathbb{R}^d,\ t \in (1,\infty).$$

If furthermore (16) is satisfied then there exist constants c and δ such that

$$p_t(x) \ge ct^{-d/\beta}, \quad |x| \le \delta t^{1/\beta},\ t \in (1,\infty). \tag{38}$$

Proof. By (1) and (13) we have

$$\begin{aligned}
p_t(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t\Phi(\xi)} d\xi \\
&\le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Phi(\xi)} d\xi \\
&\le (2\pi)^{-d} \left(\int_{|\xi|\le 1} e^{-ct|\xi|^\beta} d\xi + \int_{|\xi|\ge 1} e^{-ct|\xi|^\alpha} d\xi \right) \\
&= (2\pi)^{-d} \left(t^{-d/\beta} \int_{|u|\le t^{1/\beta}} e^{-c|u|^\beta} du + t^{-d/\alpha} \int_{|u|\ge t^{1/\alpha}} e^{-c|u|^\alpha} du \right) \\
&\le c\left(t^{-d/\beta} + t^{-d/\alpha}\right) \\
&\le ct^{-d/\beta}, \quad x \in \mathbb{R}^d,\ t \in (1,\infty).
\end{aligned}$$

We omit the proof of (38) since it is analogous to the proof of Lemma 3. $\square$

Proof of the Theorem 1. Similarly like in the proof of [18, Theorem 3] and [2, Lemma 6] for $m = \gamma+\alpha+\eta+1$ by Lemma 2, Corollary 6 and (6) we obtain

$$\begin{aligned}
p_t(x) &= \int_{\mathbb{R}^d} \tilde{p}_t(x-z)\bar{P}_t(dz) \\
&\leq c\int_{\mathbb{R}^d} t^{-d/\alpha}(1+t^{-1/\alpha}|x-z|)^{-m}\bar{P}_t(dz) \\
&= ct^{-d/\alpha}\int_0^1 \bar{P}_t(\{z\,:\,(1+t^{-1/\alpha}|x-z|)^{-m} > s\})ds \\
&\leq ct^{-d/\alpha}\int_0^1 \bar{P}_t(B(x, t^{1/\alpha}s^{-1/m}))ds \\
&\leq ct^{-d/\alpha}\int_0^1 \frac{t^{1+\gamma/\alpha}}{\bar{q}(t^{1/\alpha})}s^{-\gamma/m}\left(1+\frac{s^{-\alpha/m}\bar{q}(t^{1/\alpha})}{\bar{q}(t^{1/\alpha}s^{-1/m})}\right)|x|^{-\alpha-\gamma}\bar{q}(|x|)ds \\
&\leq c\frac{t^{1+\frac{\gamma-d}{\alpha}}}{\bar{q}(1)}|x|^{-\alpha-\gamma}\bar{q}(|x|)\left[\int_0^1 s^{-\gamma/m}ds + \int_0^1 s^{-(\gamma+\alpha+\eta)/m}ds\right] \\
&= ct^{1+\frac{\gamma-d}{\alpha}}|x|^{-\alpha-\gamma}\bar{q}(|x|), \quad x\in\mathbb{R}^d,\ t\in(0,1).
\end{aligned}$$

This, together with Lemma 7 gives (5). $\square$

LEMMA 9 *If (2), (3), (4) and (12) hold then there exists a constant c such that*

$$\bar{q}(s) \leq cs^{\alpha-\beta}, \quad s\geq 1, \tag{39}$$

and

$$\int_{|y|<r} |y|^2\nu(dy) \leq cr^{2-\beta}, \quad r\geq 1. \tag{40}$$

Proof. If $\beta=\alpha$ then (39) follows from the boundedness of $\bar{q}$. For $\beta>\alpha$ by (12) we have

$$\begin{aligned}
\infty > \int_1^\infty u^{\beta-\alpha-1}\bar{q}(u)du &\geq \int_1^s u^{\beta-\alpha-1}\bar{q}(u)du \\
&\geq \bar{q}(s)\int_1^s u^{\beta-\alpha-1}du \\
&\geq c\bar{q}(s)s^{\beta-\alpha}, \quad s\geq 2.
\end{aligned}$$

This and the boundedness of $\bar{q}$ yield (39).
By (8) we have

$$\begin{aligned}\int_{|y|<r}|y|^2\nu(dy) &\leq \int_{|y|<1}|y|^2\nu(dy)+\sum_{j=0}^{\lfloor\log_2 r\rfloor}\int_{2^j\leq|y|<2^{j+1}}|y|^2\nu(dy)\\ &\leq c_1+\sum_{j=0}^{\lfloor\log_2 r\rfloor}2^{2(j+1)}\nu(B(0,2^j)^c)\\ &\leq c_1+c_2\sum_{j=0}^{\lfloor\log_2 r\rfloor}2^{2j}2^{-j\alpha}\bar{q}(2^j).\end{aligned}$$

If $\beta<2$ then (40) follows from (39). If $\beta=2$ then using (3) and (12) we obtain

$$\begin{aligned}\sum_{j=0}^{\lfloor\log_2 r\rfloor}\left(2^j\right)^{2-\alpha}\bar{q}(2^j) &\leq \sum_{j=0}^{\infty}\left(2^j\right)^{2-\alpha}\bar{q}(2^j)\\ &\leq c\sum_{j=0}^{\infty}\int_{2^j}^{2^{j+1}}s^{1-\alpha}\bar{q}(s)ds\\ &= c\int_1^{\infty}s^{1-\alpha}\bar{q}(s)ds=c<\infty\end{aligned}$$

$\square$

The proof of Theorem 3 is similar to the proof of Theorem 1 and therefore we abbreviate below the details.

Proof of the Theorem 3. Let $\tilde{P}^*_t=\tilde{P}^{t^{1/\beta}}_t$, $\tilde{p}^*_t=\tilde{p}^{t^{1/\beta}}_t$, and $\bar{P}^*_t=\bar{P}^{t^{1/\beta}}_t$. By (40) for $l\in\mathbb{N}$ and $r>1$ we have

$$\begin{aligned}\int_{\mathbb{R}^d}y_k^{2l}\tilde{\nu}_r(dy) &\leq \int_{|y|<r}|y|^{2l}\nu(dy)\\ &\leq r^{2l-2}\int_{|y|<r}|y|^2\nu(dy)\leq cr^{2l-\beta},\end{aligned}$$

and (25) yields

$$\int_{\mathbb{R}^d}|y|^{2n}\tilde{P}^r_t(dy) \leq r^{2n}d^{n-1}\sum_{k=1}^{d}\sum_{j=1}^{2n}c(j,n)t^jr^{-j\beta}.$$

Taking $r = t^{1/\beta}$ we get

$$\int_{\mathbb{R}^d} |y|^{2n} \tilde{P}_t^*(dy) \le c(n) t^{2n/\beta}, \quad t \ge 1. \tag{41}$$

From (8) and (39) we obtain $\nu(B(0,r)^c) \le cr^{-\beta}$ for $r \ge 1$, and similarly like in the proof of Lemma 2, using (13), (19) and (41), for every $m \ge 1$ we get

$$\tilde{p}_t^*(y) \le c(m) t^{-d/\beta}(1 + t^{-1/\beta}|y|)^{-m}, \quad y \in \mathbb{R}^d,\, t \in (1,\infty).$$

By Corollary 5 and (39) we get

$$\bar{P}_t^*(B(x,r)) \le ctr^\gamma \left(1 + \frac{t^{-\alpha/\beta}\bar{q}(t^{1/\beta})}{r^{-\alpha}\bar{q}(r)}\right) |x|^{-\alpha-\gamma}\bar{q}(|x|)\,, \quad x \in \mathbb{R}^d,\, t \ge 1,\, r > 0,$$

and for $m = \gamma + \alpha + \eta + 1$ we obtain

$$\begin{aligned}
p_t(x) &= \int_{\mathbb{R}^d} \tilde{p}_t^*(x-z)\bar{P}_t^*(dz) \\
&\le c\int_{\mathbb{R}^d} t^{-d/\beta}(1 + t^{-1/\beta}|x-z|)^{-m}\bar{P}_t^*(dz) \\
&\le ct^{-d/\beta}\int_0^1 \bar{P}_t^*(B(x, t^{1/\beta}s^{-1/m}))ds \\
&\le ct^{-d/\beta}\int_0^1 t^{1+\gamma/\beta}s^{-\gamma/m}\left(1 + \frac{s^{-\alpha/m}\bar{q}(t^{1/\beta})}{\bar{q}(t^{1/\beta}s^{-1/m})}\right)|x|^{-\alpha-\gamma}\bar{q}(|x|)ds \\
&\le ct^{1+\frac{\gamma-d}{\beta}}|x|^{-\alpha-\gamma}\bar{q}(|x|)\left[\int_0^1 s^{-\gamma/m}ds + c\int_0^1 s^{-(\gamma+\alpha+\eta)/m}ds\right] \\
&= ct^{1+\frac{\gamma-d}{\beta}}|x|^{-\alpha-\gamma}\bar{q}(|x|), \quad x \in \mathbb{R}^d,\, t \in (1,\infty).
\end{aligned}$$

We get (14) by this and the Lemma 8. $\square$

Proof of the Theorem 2. Let $a \in (0,1)$ and $t \in (0,1)$. For $|x| > r + at^{1/\alpha}$ by (32) and (10) we get

$$\bar{P}_t^{at^{1/\alpha}}(B(x,r)) \ge e^{-ca^{-\alpha}}t\bar{\nu}_{at^{1/\alpha}}(B(x,r)) = e^{-ca^{-\alpha}}t\nu(B(x,r)). \tag{42}$$

This, Lemma 3 and (9) for $x \in A$ yield

$$p_t(x) = \tilde{p}_t^{at^{1/\alpha}} * \bar{P}_t^{at^{1/\alpha}}(x)$$

$$
\begin{aligned}
&= \int \tilde{p}_t^{at^{1/\alpha}}(x-z)\bar{P}_t^{at^{1/\alpha}}(dz) \\
&\geq c\int_{|z-x|<c_1e^{-c_2a^{-\alpha}}t^{1/\alpha}} t^{-d/\alpha}\bar{P}_t^{at^{1/\alpha}}(dz) \\
&= ct^{-d/\alpha}\bar{P}_t^{at^{1/\alpha}}(B(x,c_1e^{-c_2a^{-\alpha}}t^{1/\alpha})) \\
&\geq c(a)t^{1+\frac{\gamma-d}{\alpha}}|x|^{-\alpha-\gamma}\underline{q}(|x|),
\end{aligned}
$$

provided $|x| > (a + c_1e^{-c_2a^{-\alpha}})t^{1/\alpha}$. By Lemma 7 we have $p_t(x) \geq ct^{-d/\alpha}$ for $|x| < \delta t^{1/\alpha}$. We choose $a \in (0,1)$ such that $a + c_1e^{-c_2a^{-\alpha}} \leq \delta$ and we obtain (11). $\square$

Proof the Theorem 4. If follows from (15) and (16) that $\tilde{p}_t^{at^{1/\beta}}(y) \geq ct^{-d/\beta}$ for $t \geq 1$, $a \in (0,1)$, and $|y| \leq c_1e^{-c_2a^{-\beta}}t^{1/\beta}$ (cf. Lemma 3). Using this and Lemma 8 we get (17) in the same way as (11) above. $\square$

5 Tempered stable processes

In what follows we assume that

$$
\nu(D) = \int_{\mathbb{S}}\int_0^\infty \mathbf{1}_D(s\theta)s^{-1-\alpha}Q(\theta,s)ds\mu(d\theta), \quad D \subset \mathbb{R}^d, \tag{43}
$$

where μ is a bounded symmetric measure on $\mathbb{S}$ and Q is a nonnegative bounded function such that $Q(-\theta,s) = Q(\theta,s)$, $\theta \in \mathbb{S}$, $s > 0$. We consider the semigroup of probability measures $\{P_t, t \geq 0\}$ with the Fourier transform $\mathcal{F}(P_t)(\xi) = \exp(-t\Phi(\xi))$, where

$$
\Phi(\xi) = \int_{\mathbb{R}^d} (1-\cos(\xi\cdot y))\,\nu(dy), \quad \xi \in \mathbb{R}^d. \tag{44}
$$

In [20] J. Rosiński investigate the *tempered stable process* with the Lévy measure given by (43) with $Q(\theta,\cdot)$ completely monotone for every $\theta \in \mathbb{S}$. We do not assume here that Q is completely monotone and consider a wider class of processes.

Lemma 10 *If there exist $s_0 \in (0,1)$ such that*

$$
\inf_{s\in(0,s_0),\eta\in\mathbb{S}} \int_{\mathbb{S}} |\eta\cdot\theta|^2 Q(\theta,s)\mu(d\theta) > 0, \tag{45}
$$

and

$$\int_{\mathbb{R}^d} |y|^2 \nu(dy) = \int_{\mathbb{S}} \int_0^\infty s^{1-\alpha} Q(\theta, s) ds \mu(d\theta) < \infty, \tag{46}$$

then

$$\Phi(\xi) \approx |\xi|^2 \wedge |\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \tag{47}$$

Proof. By (46) we have

$$\Phi(\xi) \le \frac{1}{2} \int_{\mathbb{R}^d} |\xi \cdot y|^2 \nu(dy) \le \frac{1}{2} |\xi|^2 \int_{\mathbb{R}^d} |y|^2 \nu(dy) = c|\xi|^2, \quad \xi \in \mathbb{R}^d.$$

For $|\xi| > 1$ by the boundedness of Q and μ we obtain also

$$\begin{aligned} \Phi(\xi) &\le \frac{1}{2} \int_{|y| \le 1/|\xi|} |\xi \cdot y|^2 \nu(dy) + 2 \int_{|y| > 1/|\xi|} \nu(dy) \\ &\le \frac{1}{2} |\xi|^2 \int_0^{1/|\xi|} \int_{\mathbb{S}} s^{1-\alpha} Q(\theta, s) ds \mu(d\theta) + 2 \int_{1/|\xi|}^\infty \int_{\mathbb{S}} s^{-1-\alpha} Q(\theta, s) ds \mu(d\theta) \\ &\le c|\xi|^\alpha. \end{aligned}$$

Moreover by (45) we get

$$\begin{aligned} \Phi(\xi) &\ge (1 - \cos 1) \int_{|y| < 1/|\xi|} |\xi \cdot y|^2 \nu(dy) \\ &= (1 - \cos 1) \int_{\mathbb{S}} \int_0^{1/|\xi|} |s\xi \cdot \theta|^2 s^{-1-\alpha} Q(\theta, s) ds \mu(d\theta) \\ &\ge c|\xi|^2 \int_0^{\frac{1}{|\xi|} \wedge s_0} s^{1-\alpha} ds, \quad \xi \in \mathbb{R}^d, \end{aligned}$$

and the Lemma follows. □

It follows from Lemma 10 that we can apply the Theorem 1 with $\bar{q}(s) \equiv$ *const.* to every Lévy measure given by (43) with Q and μ satisfying (4) and (45). In specific cases we can obtain certainly more precise results.

We call a measure λ *degenerate* if there is a proper linear subspace M of $\mathbb{R}^d$ such that $\text{supp}(\lambda) \subset M$; otherwise we call λ *nondegenerate*. Note that if μ is nondegenerate and $Q(\theta, s) \ge c > 0$ on $\text{supp}(\mu) \times (0, s_0)$ for some $s_0 > 0$ then (45) is satisfied.

The following theorem follows from Lemma 10 and the Theorems 1, 2, 3 and 4 with $\beta = 2$ and $\bar{q}(s) = \underline{q}(s) = (1+s)^{-m}$. We omit the easy proof.

THEOREM 5 *If there exists $m > 2-\alpha$ such that*

$$Q(\theta, s) \approx (1+s)^{-m}, \quad s > 0,\, \theta \in \mathbb{S}, \tag{48}$$

and c and $\gamma \in [1, d]$ such that

$$\mu(B(\theta, r) \cap \mathbb{S}) \le c r^{\gamma-1}, \quad r < 1/2,\, \theta \in \mathbb{S}, \tag{49}$$

and μ is nondegenerate then

$$p_t(x) \le C \min\left(t^{-d/\alpha}, \frac{t^{1+\frac{\gamma-d}{\alpha}}}{|x|^{\alpha+\gamma}(1+|x|)^m} \right), \quad x \in \mathbb{R}^d,\, t \in (0,1),$$

and

$$p_t(x) \le C \min\left(t^{-d/2}, \frac{t^{1+\frac{\gamma-d}{2}}}{|x|^{\alpha+\gamma}(1+|x|)^m} \right), \quad x \in \mathbb{R}^d,\, t \in (1,\infty).$$

If additionally there exist a constant c and a set $A_0 \subset \mathbb{S}$ such that

$$\mu(B(\theta, r) \cap \mathbb{S}) \ge c r^{\gamma-1}, \quad \theta \in A_0,\, r < 1/2, \tag{50}$$

then

$$p_t(x) \ge C \min\left(t^{-d/\alpha}, \frac{t^{1+\frac{\gamma-d}{\alpha}}}{|x|^{\alpha+\gamma}(1+|x|)^m} \right), \quad x \in A,\, t \in (0,1),$$

and

$$p_t(x) \ge C \min\left(t^{-d/2}, \frac{t^{1+\frac{\gamma-d}{2}}}{|x|^{\alpha+\gamma}(1+|x|)^m} \right), \quad x \in A,\, t \in (1,\infty),$$

where $A = \{r\theta : \, r > 0,\, \theta \in A_0\}$.

If μ is absolutely continuous with respect to the standard surface measure on $\mathbb{S}$ and its density g_μ is such that $c^{-1} \le g_\mu(\theta) \le c$, $\theta \in \mathbb{S}$, for some constant $c > 0$ then we have $\mu(B(\theta, r)) \approx r^{d-1}$, $\theta \in \mathbb{S}$, $r \le 1/2$. Therefore in this case for $Q(\theta, s)$ satisfying (48) we obtain

$$p_t(x) \approx \min\left(t^{-d/\alpha}, \frac{t}{|x|^{\alpha+d}(1+|x|)^m} \right), \quad x \in \mathbb{R}^d,\, t \in (0,1),$$

and

$$p_t(x) \approx \min\left(t^{-d/2}, \frac{t}{|x|^{\alpha+d}(1+|x|)^m}\right), \quad x \in \mathbb{R}^d,\, t \in (1,\infty).$$

We can also apply our results for ν given by (43), nondegenerate μ satisfying (49) and Q such that

$$c^{-1}(1+s)^a e^{-c_1 s} \le Q(\theta, s) \le c(1+s)^a e^{-c_2 s}, \quad s>0,\, \theta \in \mathbb{S}, \tag{51}$$

for some $a \ge 0$. In this case for every $m>0$ we also have

$$Q(\theta,s) \le c_m(1+s)^{-m}, \quad s>0,\, \theta\in\mathbb{S}, \tag{52}$$

and therefore from Lemma 10 and Theorems 1 and 3 we obtain

$$p_t(x) \le C_m \min\left(t^{-d/\alpha}, \frac{t^{1+\frac{\gamma-d}{\alpha}}}{|x|^{\alpha+\gamma}(1+|x|)^m}\right), \quad x\in\mathbb{R}^d,\, t\in(0,1), \tag{53}$$

and

$$p_t(y) \le C_m \min\left(t^{-d/2}, \frac{t^{1+\frac{\gamma-d}{2}}}{|x|^{\alpha+\gamma}(1+|x|)^m}\right), \quad x\in\mathbb{R}^d,\, t\in(1,\infty). \tag{54}$$

for every $m > 2-\alpha$. If we assume additionally (50) then by Theorems 2 and 4 with $\underline{q}(s) = (1+s)^a e^{-2c_2 s}$ we get

$$p_t(x) \ge C \min\left(t^{-d/\alpha}, \frac{t^{1+\frac{\gamma-d}{\alpha}}}{|x|^{\alpha+\gamma}}\,(1+|x|)^a e^{-c_3|x|}\right), \quad x\in A,\, t\in(0,1), \tag{55}$$

and

$$p_t(x) \ge C \min\left(t^{-d/2}, \frac{t^{1+\frac{\gamma-d}{2}}}{|x|^{\alpha+\gamma}}\,(1+|x|)^a e^{-c_3|x|}\right), \quad x\in A,\, t\in(1,\infty). \tag{56}$$

We like to discuss the particular case of the relativistic α–stable Lévy process with $\Phi_m(\xi) = (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ for $m>0$ which is investigated, e.g., in [21], [16] and [11]. We consider here only $m=1$ because

$$p_t^m(x) = m^{d/\alpha} p^1_{mt}(m^{1/\alpha}x),$$

where p_t^m denotes the transition densities corresponding to Φ_m, see [11]. The Lévy measure in this case has the form

$$\begin{aligned}\nu(D) &= c_1 \int_D |y|^{-d-\alpha} K_{d,\alpha}(|y|) dy \\ &= c_2 \int_{\mathbb{S}} \int_0^\infty \mathbf{1}_D(s\theta) s^{-1-\alpha} K_{d,\alpha}(s) ds \sigma(d\theta), \quad D \subset \mathbb{R}^d,\end{aligned}$$

where σ is the standard isotropic surface measure on $\mathbb{S}$ and

$$K_{d,\alpha}(s) = s^{d+\alpha} \int_0^\infty e^{-u} e^{-\frac{s^2}{4u}} u^{\frac{-2-d-\alpha}{2}} du, \quad s > 0.$$

We have (see [11])

$$K_{d,\alpha}(s) \approx (1+s)^{\frac{d+\alpha-1}{2}} e^{-s}.$$

This yields that (51) for $Q(\theta, s) = K_{d,\alpha}(s)$ and $a = \frac{d+\alpha-1}{2}$ holds and we obtain the estimates (53), (54), (55) and (56) with $\gamma = d$, $A = \mathbb{R}^d$, and $a = \frac{d+\alpha-1}{2}$. The sharp estimates of the transition densities of the relativistic process are given also in [10].

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